

# The Regularized Trace of Sturm-Liouville Problem with Discontinuities at Two Points

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## Abstract

In this paper, we obtain a regularized trace formula for a Sturm Liouville problem which has two points of discontinuity and also contains an eigenparameter in a boundary condition.

## 1 Introduction

The regularized trace for the classical Sturm Liouville problem was first calculated by Gelfand and Levitan [19]. Then this work was continued by many authors ( see [1], [5-8], [12-18] and [20], respectively). A regularized trace formula for Sturm-Liouville equation with one or two boundary conditions depending on a spectral parameter was investigated in [2,4,6,11,22,24]. The regularized trace formula of the infinite sequence of eigenvalues for some version of a Dirichlet boundary value problem with turning points was calculated in [9]. All of these works are on the traces of continuous boundary value problems.

Consider the boundary value problem

$$\tau(u) := -u'' + q(x)u = \lambda u, \quad x \in I, \quad (1.1)$$

with boundary conditions

$$B_a(u) := u'(a) = 0, \quad (1.2)$$

$$B_b(u) := (\lambda - h)u(b) - u'(b) = 0, \quad (1.3)$$

and transmission conditions at two points of discontinuity

$$T_{c_1}(u) := \begin{pmatrix} u(c_1+) \\ u'(c_1+) \end{pmatrix} - \frac{1}{\delta} \begin{pmatrix} u(c_1-) \\ u'(c_1-) \end{pmatrix} = 0, \quad (1.4)$$

$$T_{c_2}(u) := \begin{pmatrix} u(c_2-) \\ u'(c_2-) \end{pmatrix} - \frac{\gamma}{\delta} \begin{pmatrix} u(c_2+) \\ u'(c_2+) \end{pmatrix} = 0, \quad (1.5)$$

where  $I := [a, c_1) \cup (c_1, c_2) \cup (c_2, b]$ ,  $\lambda$  is a spectral parameter,  $q(x)$  is a real valued function which is continuous in  $[a, c_1)$ ,  $(c_1, c_2)$  and  $(c_2, b]$  and has finite limits  $q(c_1 \pm) := \lim_{x \rightarrow c_1 \pm} q(x)$ ,  $q(c_2 \pm) := \lim_{x \rightarrow c_2 \pm} q(x)$ ,  $h, \delta, \gamma$  are real numbers  $\delta \neq 0, \gamma \neq 0$ .

As far as we know, there are two works about the regularized trace of discontinuous boundary value problems [21,23]. They are investigated by Yang. In [21] and [23], the author studied regularized sums from the eigenvalues, oscillations of eigenfunctions and the solutions of inverse nodal problem for a discontinuous boundary value problem with retarded argument and obtained some formulas for the regularized traces of second-order differential operators with discontinuities inside a finite interval, respectively. In these works, the problem has one point of discontinuity and not contain a spectral parameter in boundary conditions.

The aim of the present paper is to obtain a formula for the regularized trace of the problem (1.1)-(1.5). The problem (1.1)-(1.5) is a Sturm Liouville problem which has two points of discontinuity and the boundary conditions depending on an eigenparameter. The regularized trace of Sturm Liouville problem with two discontinuities has not been studied before. We follow the method for computing the traces of the problem (1.1)-(1.5) in [16,17]. Firstly, we give some preliminaries for asymptotic formulas of solution and eigenvalues. Then, we proved the regularized first trace formula for the problem (1.1)-(1.5).

## 2 Preliminaries

Let us denote the solution of equation (1.1) by

$$\phi(x, \lambda) = \begin{cases} \phi_1(x, \lambda), & x \in [a, c_1), \\ \phi_2(x, \lambda), & x \in (c_1, c_2), \\ \phi_3(x, \lambda), & x \in (c_2, b], \end{cases} \quad (2.1)$$

satisfying the initial conditions

$$\phi_1(a, \lambda) = 1, \quad \phi_1'(a, \lambda) = 0, \quad (2.2)$$

$$\phi_2(c_1, \lambda) = \delta^{-1} \phi_1(c_1-, \lambda), \quad \phi_2'(c_1, \lambda) = \delta^{-1} \phi_1'(c_1-, \lambda), \quad (2.3)$$

$$\phi_3(c_2, \lambda) = \delta \gamma^{-1} \phi_2(c_2-, \lambda), \quad \phi_3'(\theta_{+\varepsilon}) = \delta \gamma^{-1} \phi_2'(c_2-, \lambda). \quad (2.4)$$

It is obvious that  $\phi(x, \lambda)$  satisfies the equation (1.1) on  $I$ , the boundary condition (1.2) and the transmission conditions (1.4) and (1.5).

The asymptotics formulas of the eigenvalues and eigenfunctions can be derived similar to the classical techniques of [3, 5, 13, 19]. We state the results briefly. Denote  $\lambda = s^2$ . The solution of equation (1.1), fulfilling the conditions

(2.2)-(2.4), satisfies the integral equation

$$\begin{aligned}
\phi_3(x, \lambda) = & \frac{1}{\gamma} \cos(s(x-a)) + \frac{1}{2s\gamma} (Q_1(c_1) + Q_2(c_2)) \sin(s(x-a)) + \\
& \frac{1}{4s^2\gamma} \{q(c_2) \cos(s(x-2c_2+a)) - (q(a) + Q_1(c_1) Q_2(c_2)) \times + \\
& \cos(s(x-a))\} + \frac{1}{8s^3\gamma} \{q(c_1) (Q_1(c_1) + Q_2(c_2)) \sin(s(x-2c_1+a)) - \\
& (q(c_2) Q_1(c_1) - q'(c_2)) \sin(s(x-2c_2+a)) - (q'(a) + q(a) Q_2(c_2)) \times + \\
& \sin(s(x-a))\} + \frac{1}{s} \int_{c_2}^x \sin(s(x-y)) q(y) \phi_3(y, \lambda) dy, \tag{2.5}
\end{aligned}$$

where

$$Q_1(x) = \int_a^x q(y) dy, \quad Q_2(x) = \int_{c_1}^x q(y) dy. \tag{2.6}$$

Solving the equation (2.5) by the method of successive approximations, we obtain

$$\begin{aligned}
\phi_3(x, \lambda) = & \frac{1}{\gamma} \cos(s(x-a)) + \frac{\alpha_1(x)}{s\gamma} \sin(s(x-a)) + \\
& \frac{\alpha_2(x)}{s^2\gamma} \cos(s(x-a)) + \frac{1}{s^3\gamma} \{\alpha_3(x) \sin(s(x-a)) + \\
& \alpha_4(x) \sin(s(x-2c_2+a))\} + O\left(\frac{1}{s^4}\right), \tag{2.7}
\end{aligned}$$

$$\begin{aligned}
\phi'_3(x, \lambda) = & -\frac{s}{\gamma} \sin(s(x-a)) + \frac{\alpha_1(x)}{\gamma} \cos(s(x-a)) + \frac{1}{s\gamma} (\alpha'_1(x) - \alpha_2(x)) \times \\
& \sin(s(x-a)) + \frac{1}{s^2\gamma} \{(\alpha_3(x) + \alpha'_3(x)) \cos(s(x-a)) + \\
& \alpha_4(x) \cos(s(x-2c_2+a))\} + \frac{1}{s^3\gamma} \{\alpha'_3(x) \sin(s(x-a)) + \\
& \alpha'_4(x) \sin(s(x-2c_2+a))\} + O\left(\frac{1}{s^4}\right), \tag{2.8}
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1(x) &= \frac{1}{2}(Q_1(c_1) + Q_2(c_2) + Q_3(x)), \\
\alpha_2(x) &= \frac{1}{4}(q(x) - q(a) - Q_1(c_1)Q_2(c_2) - Q_3(x)(Q_1(c_1) + Q_2(c_2))), \\
\alpha_3(x) &= \frac{1}{8}(q(x)(Q_1(c_1) + Q_2(c_2)) - q'(x) - q'(a) - q(a)Q_2(c_2) + \\
&\quad q(c_1)(Q_1(c_1) + Q_2(c_2)) - Q_3(x)(q(a) + Q_1(c_1)Q_2(c_2))), \\
\alpha_4(x) &= \frac{1}{4}(q(c_2)Q_1(c_1) - q'(c_2)) + \frac{1}{8}q(c_2)(Q_3(x) + Q_2(c_2)), \\
Q_3(x) &= \int_{c_2}^x q(y) dy.
\end{aligned} \tag{2.9}$$

It is obvious that the characteristic function  $\omega(\lambda)$  of the problem (1.1)-(1.5) is as follows

$$\omega(\lambda) = (\lambda - h)\phi_3(b, \lambda) - \phi_3'(b, \lambda), \tag{2.10}$$

and the eigenvalues of the problem (1.1)-(1.5) coincide with the roots of  $\omega(\lambda)$ . From (2.7) and (2.8) we have

$$\begin{aligned}
\omega(\lambda) &= \frac{s^2}{\gamma} \cos(s(b-a)) + \frac{s}{\gamma} (1 + \alpha_1(b)) \sin(s(b-a)) + \\
&\quad \frac{1}{\gamma} (\alpha_2(b) - \alpha_1(b) - h) \cos(s(b-a)) + \frac{1}{s\gamma} \{(\alpha_3(b) - h\alpha_1(b) + \\
&\quad \alpha_2(b) - \alpha_1'(b)) \sin(s(b-a)) + \alpha_4(b) \sin(s(b-2c_2+a))\} + \\
&\quad O\left(\frac{1}{s^2}\right).
\end{aligned} \tag{2.11}$$

By the Rouché theorem and from the asymptotic formula (2.11), we obtain

$$s_n = \frac{(n-1/2)\pi}{b-a} + \frac{1}{(n-1/2)\pi} (1 + \alpha_1(b)) + O\left(\frac{1}{n^2}\right). \tag{2.12}$$

It follows from (2.12) that

$$\lambda_n = \left(\frac{(n-1/2)\pi}{b-a}\right)^2 + K + O\left(\frac{1}{n^2}\right), \tag{2.13}$$

where

$$K = \frac{2}{b-a} (1 + \alpha_1(b)), \tag{2.14}$$

$$\alpha_1(b) = \frac{1}{2}(Q_1(c_1) + Q_2(c_2) + Q_3(b)), \tag{2.15}$$

$$Q_1(c_1) = \int_a^{c_1} q(y) dy, \quad Q_2(c_2) = \int_{c_1}^{c_2} q(y) dy, \quad Q_3(b) = \int_{c_2}^b q(y) dy. \tag{2.16}$$

### 3 Traces of The Problem

The series

$$\lambda_0 + \sum_{n=1}^{\infty} \left( \lambda_n - \left( \frac{(n-1/2)\pi}{b-a} \right)^2 - K \right) < \infty \quad (3.1)$$

converges and is called the regularized first trace for the problem (1.1)-(1.5). The goal of this paper is to prove its sum. Our proof is based on the works of [16,17].

**Theorem 1** *Suppose that  $q(x)$  has a second-order piecewise integrable derivatives on  $[a, b]$ , then the following regularized trace formula holds*

$$\begin{aligned} s_\lambda &= \sum_{n=0}^{\infty} \left( \lambda_n - \left( \frac{(n-1/2)\pi}{b-a} \right)^2 - K \right) \\ &= h - \frac{1}{2} - \frac{2}{b-a} - \frac{\pi^2}{4(b-a)^2} - \frac{1}{4}(q(b) - q(a)) - \\ &\quad \frac{1}{b-a} (Q_1(c_1) + Q_2(c_2) + Q_3(b)) - \\ &\quad \frac{1}{8} (Q_1^2(c_1) + Q_2^2(c_2) + Q_3^2(b)), \end{aligned} \quad (3.2)$$

where  $K, \alpha_1(b)$  and  $Q_i(c_i)$  ( $i = 1, 2, 3$ ) satisfies the equations (2.14)-(2.16).

**Proof.** ...

**Lemma 2** *If  $\left| \left( \frac{(n-1/2)\pi}{b-a} \right)^2 - \lambda_n \right| \leq \rho$ , then*

$$\sum_{n=1}^{\infty} \frac{\left| \left( \frac{(n-1/2)\pi}{b-a} \right)^2 - \lambda_n \right|^k}{\left( \mu^2 + \left( \frac{(n-1/2)\pi}{b-a} \right)^2 \right)^k} \leq \rho^k \frac{b-a}{2} \frac{1}{\mu^{2k-1}}. \quad (3.9)$$

■

**Proof.** This lemma can be proved similar to proof of Lemma 5.6.1 in [17, Ch5].

Substituting (2.9), (3.8), (3.14) and (3.20) into (3.23) we obtain

$$\begin{aligned} s_\lambda &= \sum_{n=0}^{\infty} \left( \lambda_n - \left( \frac{(n-1/2)\pi}{b-a} \right)^2 - K \right) \\ &= h - \frac{1}{2} - \frac{2}{b-a} - \frac{\pi^2}{4(b-a)^2} - \frac{1}{4}(q(b) - q(a)) - \\ &\quad \frac{1}{b-a} (Q_1(c_1) + Q_2(c_2) + Q_3(b)) - \frac{1}{8} (Q_1^2(c_1) + Q_2^2(c_2) + Q_3^2(b)) \end{aligned}$$

where

$$Q_1(c_1) = \int_a^{c_1} q(y) dy, \quad Q_2(c_2) = \int_{c_1}^{c_2} q(y) dy, \quad Q_3(b) = \int_{c_2}^b q(y) dy$$

completing the proof of Theorem 1.

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